Maximal mean-field solutions in the random field Ising model: the pattern of the symmetry breaking

Maximal mean-field solutions in the random field Ising model: the pattern of the symmetry breaking

Marco Guagnelli†‖, Enzo Marinari†‡¶ and Giorgio Parisi§#

† Dipartimento di Fisica and INFN, Università di Roma 'Tor Vergata', Viale della Ricerca Scientifica, 00173 Roma, Italy
‡ Department of Physics and NPAC, Syracuse University Syracuse, NY 13244, USA
§ Dipartimento di Fisica and INFN, Università di Roma 'La Sapienza', P. Aldo Moro 2, 00185 Roma, Italy

Abstract. We study the mean-field equations for the 3D random field Ising model. We discuss the phase diagram of the model, and we address the problem of finding if such equations admit more than one solution. We find two different critical values of the temperature T: one where the magnetization takes a non-zero expectation value, and one where we start to have more than one solution to the mean-field equation. We find that, inside a given solution, there are no divergent correlation lengths.

1. Introduction

The random field Ising model (RFIM) (see for example [1–3]) is waiting for pieces of new understanding and further clarifications of the relevant physical mechanisms.

Let us start by sketching the theoretical situation. For a certain time it was hoped that dimensional reduction could be the appropriate method of computing the critical behaviour of a ferromagnet in the presence of a random magnetic field. It was proven in [4] that in perturbation theory the sum of the most divergent diagrams close to the phase transition for a random field model in dimension D coincides with that of a ferromagnetic theory, without random field, in the reduced dimension $d = D - 2$. The terms that are neglected are less singular than the leading ones by a factor ξ^{-2}, ξ being as usual the correlation length. This result suggests that all the exponents of the random field system coincide with those of the corresponding ferromagnetic system in $D - 2$ dimensions.

Clearly this result cannot be correct. Simple physical arguments (confirmed by a rigorous analysis [5]) lead to the conclusion that the lower critical dimension is 2, not 3, as implied by dimensional analysis. The deep reason for this failure can be found following the non-perturbative analysis of [3,6]. Let us summarize the main results.

We assume that the system is described by the following Hamiltonian density, which is a functional of the order parameter $\phi(x)$

$$H[\phi] = \int \mathrm{d}^D x \left(\tfrac{1}{2}(\partial\phi)^2 + V(\phi) - h(x)\phi(x) \right) \tag{1}$$

‖ E-mail address: guagnelli@roma2.infn.it
¶ E-mail address: marinari@roma1.infn.it
# E-mail address: parisi@roma1.infn.it

where the random field $h(x)$ is Gaussian uncorrelated white noise with variance $g\delta(x - y)$, and g parametrizes the strength of the random field.

The stationary points of H can be found by solving the corresponding mean-field equations

$$- \Delta\phi + V'(\phi) = h(x). \tag{2}$$

When these equations admit only one solution, as happens for sufficiently large temperature, it is natural to introduce the correlation functions

$$C(x) \equiv \overline{\phi(x)\phi(0)}$$

$$G(x) \equiv \overline{\frac{\delta\phi(x)}{\delta h(0)}} = \overline{\left\langle x \left| \frac{1}{\Delta + V''(\phi)} \right| 0 \right\rangle} \tag{3}$$

where by the long bar we denote the average over all the realizations of the random magnetic fields.

These two correlation functions are the mean-field approximation to $\overline{\langle\phi(x)\rangle\langle\phi(0)\rangle}$ and $\overline{\langle\phi(x)\phi(0)\rangle_c}$ respectively. One then finds that $C(x)$ is proportional to the same correlation function of the pure system in dimensions $d = D - 2$. The functions $G(x)$ and $C(x)$ are related one to the other, in Fourier space one finds that

$$C(k) = g \int d\mu \, \rho(\mu) \frac{1}{(k^2 + \mu^2)^2}$$

$$G(k) = \int d\mu \, \rho(\mu) \frac{1}{k^2 + \mu^2}. \tag{4}$$

The function $G(k)$ is the same as for the pure system in dimension $d = D - 2$ (dimensional reduction works in configuration space with the function C, and in momentum space with the function G).

When (2) admits more than one solution we must assign a weight to each solution in order to compute expectation values. This makes life more complicated. If we label by α different solutions, and by w_α the relative weight we can write

$$C(x) = \sum_\alpha w_\alpha \overline{\phi(x)\phi(0)}$$

$$G(x) = \sum_\alpha w_\alpha \overline{\frac{\delta\phi_\alpha(x)}{\delta h(0)}} \tag{5}$$

$$= \sum_\alpha w_\alpha \overline{\left\langle x \left| \frac{1}{-\Delta + V''(\phi)} \right| y \right\rangle}.$$

By using different prescriptions for the weights w_α we can obtain different results. This is especially true if the number of different solutions of the mean-field equations increases with the volume.

Dimensional reduction can still hold, but with a crazy choice of the weights

$$w_\alpha = \frac{\text{sign} \det[-\Delta + V''(\phi_\alpha)]}{Z_w} \tag{6}$$

where Z_w is such that $\sum_\alpha w_\alpha = 1$. Here the Morse theorem states that $Z_w = 1$.

A physically motivated choice would be

$$w_\alpha = \frac{\text{sign} \det[-\Delta + V''(\phi_\alpha)] \exp(-\beta H[\phi_\alpha])}{Z_w} \qquad (7)$$

where α runs over all the solutions of the mean-field equations; minima, maxima and saddle points all together. The strange looking unusual factor $\text{sign} \det[-\Delta + V''(\phi_\alpha)]$ is needed to keep the continuity of Z_w when new solutions appear.

It could be argued that the energies of the minima are so much smaller than the energies of the saddle points and maxima that we can simply write

$$w_\alpha = \frac{\exp(-\beta H[\phi_\alpha])}{Z_w} \qquad (8)$$

and keep the sum restricted only to the minima. In the rest of this paper we follow this second strategy.

It is possible that this modified mean-field theory gives the correct results (as is implicit in the work of [7]) and that the failure of dimensional reduction is simply related to the existence of many solutions with different energy [3, 6, 8, 9].

Our aim is here to investigate numerically this improved mean-field approximation, to make its predictions explicit, and eventually to compare them with Monte Carlo simulations. We have been motivated to start this investigation by an interesting paper [10], in which it was suggested that replica symmetry is already broken at the point ferromagnetic phase transition. For results obtained both in the mean-field framework and with a Monte Carlo and a $T = 0$ optimization approach, see [7, 11, 12].

In this paper we limit ourselves to the study of two particular solutions of the mean-fields equations, which we call ϕ_+ and ϕ_-. They are such that for any solution ϕ_α (and for any x) the relation $\phi_-(x) \leqslant \phi_\alpha(x) \leqslant \phi_+(x)$ holds. The existence of two solutions with this property (in the high-temperature phase they coincide) follows from convexity arguments [13]. We call them *maximal mean-field solutions*.

2. Lattice mean-field equations

We consider the random field Ising model (RFIM) with Ising type (Z_2) variables defined on a 3D simple cubic lattice. We study the solutions of its mean-field equations.

With i we denote the triplet of integers (x, y, z), which characterize the lattice sites. We will consider configurations of the random field $\{h_i \equiv \theta_i \mathcal{H}\}$, where the quenched random variables θ_i can take the values ± 1 with probability $\frac{1}{2}$, and we have chosen the absolute value of the field, $\mathcal{H}$, to be 1.5. Such a choice for $\mathcal{H}$ was meant to allow the critical temperature T_c to have a non-negligible shift from T_c in the pure model, and simultaneously not to be large enough to allow the transition to become first order [1].

In the mean-field approximation one introduces local magnetization variables m_i, which play the same role as $\phi(x)$ in the continuum formalism. The total free energy is written as a function of the local magnetization, and the condition for the free energy being stationary is the usual mean-field equation

$$m_i = \tanh(\beta(\mathcal{D}m_i + h_i)) \qquad (9)$$

where with $\mathcal{D}m_i$ we define the lattice sum over the six first-neighbour variables.

If this equation admits only one solution there is no ambiguity. If, on the other hand, there are many solutions, one has to weight (according to the previous discussion) different solutions with a weight proportional to the exponential of minus the free energy (multiplied by β).

Our ideal goal is to look for all solutions of this equation, which correspond to local minima of the free energy, but this is an awful task when the number of solutions is very large, as happens at low T. Here we have just looked for the solutions with higher, positive and negative, magnetization, (m_+ and m_-) using a simple iterative scheme. We have started the iterative procedure used to solve (9) from the two initial conditions $m_i = m_s$ and $m_i = -m_s$. Although a completely safe procedure would start from $m_s = 1$, it is more convenient (and it does not change the results) to take a value for m_s slightly smaller than one. The appropriate value of m_s depends on the temperature; in our simulations we have taken $m_s = 0.6$.

In the high T regime both these runs converge to the same (unique) solution. In a broken phase they will tend to different solutions with average magnetization of opposite signs. This procedure should be good enough to localize the temperature T below which the solution of (9) is not unique, and to give relevant quantitative hints about the structure of the phase transition.

We label the solutions of the mean-field equations, in a given realization of the magnetic field, by the index α; given the pattern of our search α is limited to take only one or two values. For each realization of the magnetic field the index α belongs to the set A (which can be, in our simulation, constituted of one or two solutions). The average over different field samples (which we denote by a bar: we denote the average over different solutions by $\langle \cdot \rangle$) is performed by having A running from 1 to N_A.

In each solution α (characterized by the $V \equiv L^3$ values of the local magnetization m_i) we compute the relevant observables. We define the total magnetization density

$$m^\alpha \equiv \frac{1}{V} \sum_i m_i^\alpha \tag{10}$$

and the sum of the squared local variables

$$q^\alpha \equiv \frac{1}{V} \sum_i (m_i^\alpha)^2. \tag{11}$$

We define the energy density

$$E^\alpha \equiv -\frac{1}{V} \sum_i (\tfrac{1}{2} m_i^\alpha \mathcal{D} m_i^\alpha + h_i m_i^\alpha) \tag{12}$$

the entropy density

$$S^\alpha \equiv -\frac{1}{V} \sum_i \left(\frac{1+m_i^\alpha}{2} \log\left(\frac{1+m_i^\alpha}{2}\right) + \frac{1-m_i^\alpha}{2} \log\left(\frac{1-m_i^\alpha}{2}\right) \right) \tag{13}$$

and the total free energy as

$$\beta F^\alpha = V(\beta E^\alpha - S^\alpha). \tag{14}$$

The weight w_α associated with each solution α is given by

$$w_\alpha = \frac{\exp(-\beta F^\alpha)}{Z_w}. \tag{15}$$

3. Numerical results for local quantities

Here we present numerical results for systems of size up to 48 in a range of β that goes from 1.1 to 1.5 (we will always give β in units of the critical β at zero random field, i.e. $\frac{1}{6}$). We have analysed 600 random field samples for the 12^3 lattice, 400 for the 24^3 lattice, 200 for the 36^3 lattice and 30 for the 48^3 lattice.

In this section we will discuss the behaviour of local quantities (i.e. those objects that can easily be constructed from the magnetization), while in the next section we will concentrate our attention on the response functions, which must be computed by inverting the lattice equivalent of $(-\Delta + V''(\phi))$, a highly non-local operation.
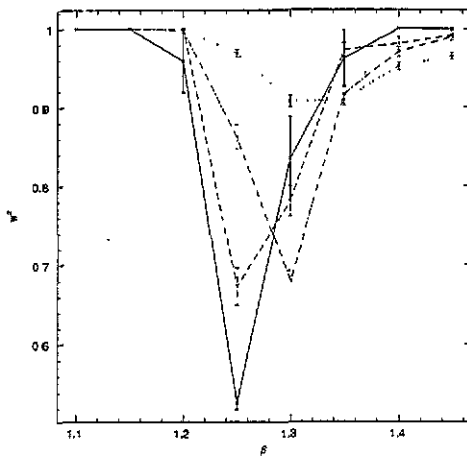

Figure 1. W^2 as a function of β. Dotted curve for the 12^3 lattice, chain curve for the 24^3 lattice, broken curve for the 36^3 lattice and full curve for the 48^3 lattice.
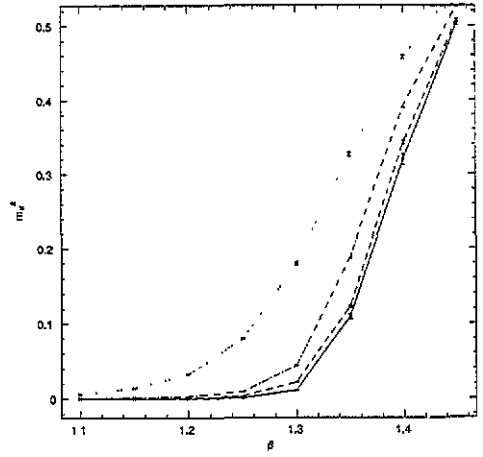
Figure 2. As in figure 1, but m_M^2.

A very interesting quantity is

$$W^2 \equiv \sum_{\alpha} w_{\alpha}^2. \tag{16}$$

This quantity is different from 1 when the mean-field equations admit more than one solution: roughly speaking W^{-2} is the average number of relevant solutions. We display the results for W^2 as a function of β in figure 1. We see that W^2 becomes significantly different from 1 only at β greater than 1.2. We see a change in regime at this beta, which we denote by β_1.

Another quantity that is interesting to measure is the maximal magnetization m_M^2, defined as $\max_{\alpha}(m^{\alpha})^2$. In figure 2 we show the β dependence of m_M^2 for different lattice sizes. We see a transition from an asymptotic zero value of m_M^2 to a non-zero value around $\beta = 1.35$. The transition becomes sharper by increasing the size of the lattice. We also see a change in regime at this new value of β, which we denote by β_2.

A more detailed understanding can be obtained by considering the correlation functions of the local magnetization. To this end we define, for each solution α, the magnetization on a 2-plane as

$$M_x^{\alpha}(\lambda) \equiv \sum_{y,z} m^{\alpha}(\lambda, y, z). \tag{17}$$

$M_y^\alpha(\lambda)$ and $M_z^\alpha(\lambda)$ are defined in an analogous way. We define the zero (bi-)momentum magnetization–magnetization correlation function for the solution α as

$$C^\alpha(\lambda) \equiv \sum_{\substack{\mu=x,y,z \\ \lambda_1,\lambda_2 \text{ such that } |\lambda_1-\lambda_2|=\lambda}} M_\mu^\alpha(\lambda_1) M_\mu^\alpha(\lambda_2). \tag{18}$$

The total correlation function at distance λ, averaged over N_A samples, is defined as

$$C(\lambda) \equiv \frac{1}{N_A} \sum_A \sum_\alpha w^\alpha C^\alpha(\lambda) \tag{19}$$

and we denote by $C_c(\lambda)$ its connected part.

At first order in perturbation theory in the strength of the random field [3,4,6] $C_c(\lambda)$ has (as we have discussed before) a double pole in Fourier space. It has also been shown that this form retains its validity at all orders in perturbation theory, and should be exact in the region where supersymmetric considerations hold. In x space this leads to

$$C(\lambda) \simeq A(1 + \lambda/\xi^{(C)}) e^{-\lambda/\xi^{(C)}} + B \tag{20}$$
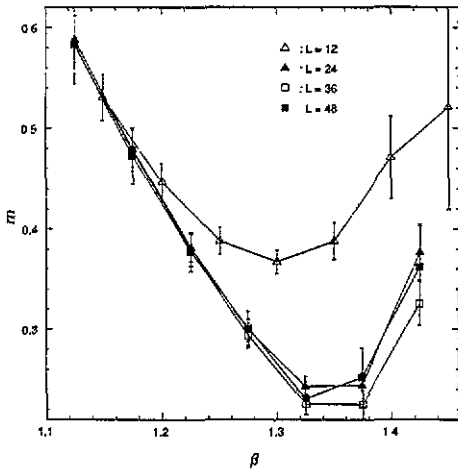
which defines the correlation length $\xi^{(C)}$.

Figure 3. The inverse correlation length $m \equiv 1/\xi^{(C)}$ as a function of β for different lattice sizes.
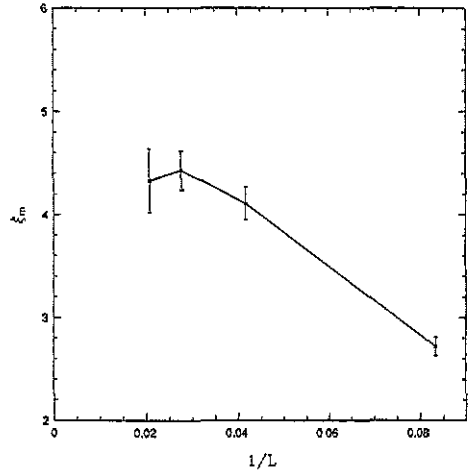
Figure 4. The maximum correlation length $\xi_m^{(C)}$ as a function of the inverse linear size of the system.

In figure 3 we plot the inverse correlation length as a function of β. We have used a global fit to $C(\lambda)$ (which has determined $\xi^{(C)}$, A and B by assuming a functional dependence that takes in account the periodic boundary conditions). In all cases we have computed the statistical errors by using a standard jack-knife procedure. We have also computed λ dependent correlation length estimators. By averaging them in the plateau region we have obtained another estimate of $\xi^{(C)}$, which turns out to be completely compatible with the one from the global fits. The fits turn out to be of very good quality, confirming the approximate validity of the form (20).
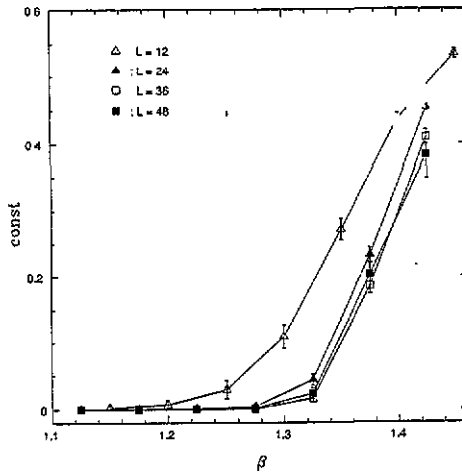
Figure 5. The constant coefficient B from the global fit to $C(\lambda)$ as a function of β.

The correlation length of figure 3 has quite a broad maximum close to $\beta = 1.35$. $\xi^{(C)}$ close to its peak increases when going from $L = 12$ to $L = 24$, but for larger lattices it remains constant.

In figure 4 we plot the maximum value of the correlation length, $\xi_m^{(C)}$, as a function of $1/L$, to stress the saturation that occurs for large L. The asymptotic $\xi_m^{(C)}$ is of order 4.5. It is rather consistent that the correlation length becomes independent of the size for sizes three to four times larger than the correlation length.

In figure 5 we plot the coefficient B (i.e. the constant asymptotic value of the correlation function) computed from the fit to $C(\lambda)$ as a function of β. For large volumes B should become identical to m^2 (which, in our analysis, turns out to be very similar to m_M^2), but its finite-size corrections are smaller, especially in the high-temperature region, where B and m_M^2 are asymptotically zero. B seems to take a non-zero expectation value starting from $\beta_2 \simeq 1.35$. This method gives a very good estimate of the value of the critical temperature where m_M^2 becomes significantly different from zero.

In the region where m_M^2 is zero all different solutions of the mean-field equations should become locally equal in the infinite volume limit, or more precisely their absolute difference should on average go to zero with the volume.

It is natural to ask if these values of β do correspond in the thermodynamic limit to real phase transitions. The quantity W^2 becomes different from zero as soon as there exists a realization of the magnetic field that admits two solutions. An explicit computation shows that if $h(i) = (-1)^{x+y+z}$ one finds two solutions when $\beta \geqslant \beta_G \simeq 1.015$†. Simple-minded arguments (which generalize the original Griffiths theorem for random diluted magnetic systems) suggest that the free energy is C^∞ but not analytic at β_G

For $\beta > \beta_G$ there exist realizations of the magnetic field for which at least two solutions exist. These field configurations are special, and their measure is small. We expect therefore that W^2 is different from 1, mathematically speaking, for $\beta \geqslant \beta_G$, but it becomes significantly different from 1 only at $\beta \geqslant \beta_1$. Similar arguments can be performed for β_2. The non-vanishing of m_M^2 for $\beta \geqslant \beta_G$ is a pathology that arises from our choice of considering only the maximal solution. If we consider the physically relevant quantity, i.e.

† An approximate formula valid for small $\mathcal{H}$ is $\beta_G = 1 + \mathcal{H}^2/144 + O(\mathcal{H}^4)$.

$$m^2 = \sum_\alpha w_\alpha (m^\alpha)^2 \tag{21}$$

it should become different from zero only at values of β much higher than β_G. The fact that the correlation length remains finite and somewhat small near β_2 may be taken as an indication that the true ferromagnetic transition at which m^2 becomes different from zero is at higher values of β.

The situation would be clarified if we could compute the full expression for $C(x)$, summing over all the solutions, but we have left this task for a future work.

4. Numerical results for the response functions

To compute the correlation functions in the mean-field approach we must use the fluctuation–dissipation theorem. We are therefore led to consider the susceptibility function $\chi_{i,j}$, which is equal to the derivative of the magnetization m_i with respect to the field h_j (for sake of typographical clarity in the following we will omit the solution label α). If there is a single stable state we have to perturb the unique solution of (9). In this way we obtain the equation

$$\chi_{i,j} = \beta(1 - m_i^2)(\mathcal{D}\chi_{i,j} + \delta_{i,j}). \tag{22}$$

This is a linear sparse equation that can be solved by using standard iterative techniques.

The computation of χ for all the values of i and j would be extremely time consuming, so we compute the Green functions $g_i \equiv \chi_{i,0}$ by setting $j = 0$ and iterating the relation

$$g_i = \beta(1 - m_i^2)(\mathcal{D}g_i + \delta_{i,0}). \tag{23}$$

We also compute the susceptibility $\chi \equiv (1/V) \sum_{i,j} \chi_{i,j}$ by iterating

$$\chi_i = \beta(1 - m_i^2)(\mathcal{D}\chi_i + 1)$$
$$\chi \equiv \frac{1}{V} \sum_{i=1}^{V} \chi_i \tag{24}$$

and the *overlap* susceptibility $\chi^q \equiv (1/V) \sum_{i,j} \chi_{i,j} m_i m_j$ from

$$\chi_i^q = \beta(1 - m_i^2)(\mathcal{D}\chi_i^q + m_i)$$
$$\chi^q \equiv \frac{1}{V} \sum_{i=1}^{V} \chi_i^q m_i. \tag{25}$$

The name *overlap susceptibility* arises from the following considerations. Let us consider two replicas (σ and τ) of the same system whose dynamics is determined by a Hamiltonian that contains the usual one system contribution plus a direct coupling between the two systems. The total Hamiltonian is

$$H[\sigma] + H[\tau] - \epsilon \sum_i \sigma_i \tau_i. \tag{26}$$

This construction is common in the study of other disordered systems like spin glasses. The quantity χ^q coincides with $\partial q / \partial \epsilon$, evaluated at $\epsilon = 0$, where q is the overlap density, i.e. $(1/V) \sum_i \sigma_i \tau_i$.

In the interesting case in which the mean-field equations admit many solutions α we follow the simplest procedure of weighting each of these with the weight w^α (recall that we are taking into account only the two maximal solutions). In this way we obtain only one term of the two that form the full susceptibility. It is easy to check that the response function

$$R(i, j) \equiv \frac{\partial}{\partial h_j} \sum_\alpha w_\alpha m_i^\alpha \tag{27}$$

is given by

$$R(i, j) = \chi_{i,j} + \beta \left(\sum_\alpha w_\alpha m_i^\alpha m_j^\alpha - \sum_\alpha w_\alpha m_i^\alpha \sum_\gamma w_\gamma m_j^\gamma \right). \tag{28}$$

The second term (in brackets), which arises in the presence of many solutions, is likely to be dominant near the critical point, as will shall see below. It may be convenient to call the first the diagonal contribution, and the second one the off-diagonal contribution.
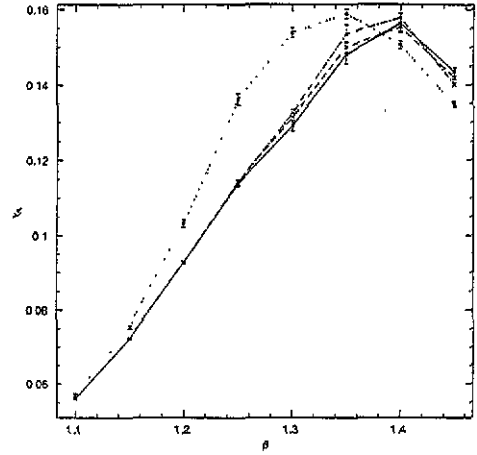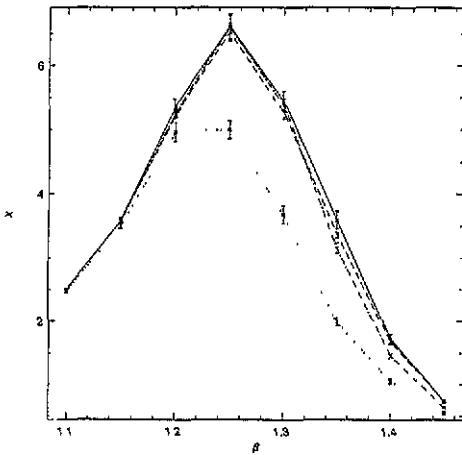

Figure 6. As in figure 1, but χ, the diagonal contribution to the susceptibility.

Figure 7. As in figure 1, but χ^q, the diagonal contribution to the overlap susceptibility.

We have computed the diagonal contributions χ and χ^q with the results shown in figures 6 and 7. It is impressive that χ has a sharp maximum close to β_1, while χ^q has a peak at much higher beta (slightly above β_2) and does not show any significant anomaly at β_1. These two peaks are volume independent for large volume. The correctness of this result is confirmed by the direct analysis of the correlation length corresponding to χ, $\xi_\chi^{(1)}$, which we show in figure 8. $\xi_\chi^{(1)}$ never becomes large in the whole region and for $\beta \leqslant \beta_1$ essentially coincides with $\xi^{(C)}$, the correlation length we have discussed in the previous section. We
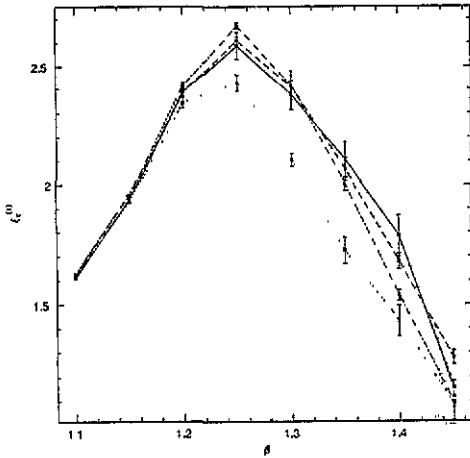
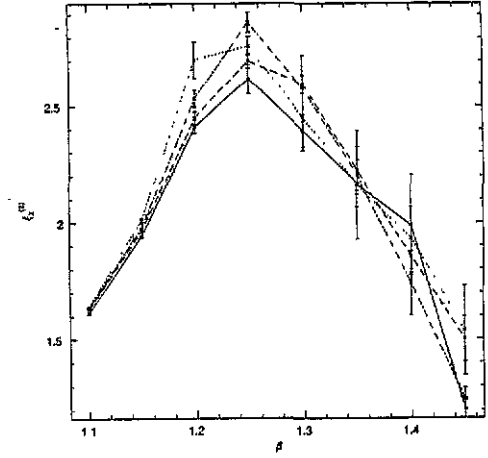**Figure 8.** As in figure 1, but $\xi_\chi^{(1)}$.   **Figure 9.** As in figure 1, but $\xi_\chi^{(2)}$.

find that the supersymmetry prediction of equality of the two correlation lengths is correct in the region $\beta \leqslant \beta_1$ where only one solution is present.

We have also considered the correlation lengths $\xi_\chi^{(n)}$ defined by taking the nth power of the zero bi-momentum correlation functions that give $\xi_\chi^{(1)}$, and by looking at their decay. They do not present a significant difference (once divided by n) from the one obtained for $n = 1$. In figure 9 we show the correlation length with $n = 2$ (which has the smallest statistical error), which can be compared with the $n = 1$ result of figure 8. The two sets of curves are very similar.

Evaluating at least some approximation to the off-diagonal contribution to the susceptibility is essential. We have done it by only using our maximal solutions. There is a large statistical error. In the low temperature region we expect that the off-diagonal contribution is proportional to $N^{1/2}$, this contribution arising from a few exceptional configurations of the magnetic field that have two solutions with opposite magnetization with similar weight. This event happens with a probability of order $1/N^{1/2}$; the corresponding off-diagonal susceptibility is of order N, so that the net contribution to the susceptibility coming from these exceptional configurations is proportional to $N^{1/2}$. In this region the off-diagonal susceptibility is dramatically increasing, showing the trend to diverge about $\beta \simeq 1.30$. Anyhow there is no convincing argument that implies that the restriction to the maximal solution should be a good approximation, apart from very close to β_1, where only two stable solutions are expected.

A full computation (including all the solutions) of both the diagonal and the non–diagonal contribution to the susceptibility would be extremely interesting.

5. Conclusions

The existence of many solutions to the mean-field equations turns out to be a crucial phenomenon; inside a single solution (at least of the maximal type) one does not see any sign of the presence of a divergent correlation length. The critical behaviour of the susceptibility and of the correlation length in a 3D RFIM is dominated by the effects of the presence of many solutions. The supersymmetric predictions start to fail exactly at the point

where one finds more than one solution of the mean-field equations. It is not surprising that dimensional reduction, which completely misses the existence of more than one solution, gives unreliable exponents at the critical point.

It is reasonable that each solution of the mean-field equation does correspond to a valley for the energy in configuration space†. In this case the dynamics of Monte Carlo simulations of a real system, also at temperature slightly above the critical one, is likely to be dominated by thermally activated tunnelling among different valleys, and we expect it to be a slow process.

Acknowledgments

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† We reserve the word state for solutions (or a cluster of solutions) such that their distance $(1/V) \sum_i (m_i^\alpha - m_i^\gamma)^2$ does not vanish in the infinite volume limit.